Large-N_c equivalence and the sign problem at finite baryon density

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Abstract

QCD with a finite baryon chemical potential, despite its importance, is not well understood because the standard lattice QCD simulation is not applicable due to the sign problem. Although QCD-like theories which do not suffer from the sign problem have been studied intensively, relation to QCD with a finite baryon chemical potential was not clear. This paper introduces large-N_c equivalences between QCD and various QCD-like theories. These equivalences lead us to a unified viewpoint for QCD with baryon and isospin chemical potentials, $SO(2N_c)$ and $Sp(2N_c)$ gauge theories, QCD with adjoint matters and two-color QCD. In particular QCD with the baryon chemical potential is equivalent to its phase quenched version in a certain parameter region, which is relevant for heavy ion collision experiments. This result is useful especially for finding or excluding a possible QCD critical point, through the phase quenched simulation of the $SU(3)$ QCD.

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1 Introduction

Consider QCD at a finite baryon chemical potential,

$$\mathcal{L} = \frac{1}{4g^2} \operatorname{tr}(F_{\mu\nu})^2 + \sum_{f=1}^{N_f} \bar{\psi}_f \left(\gamma^\mu D_\mu + m_f + \mu\gamma^4 \right) \psi_f, \tag{1}$$

where the gauge group is $SU(3)$, N_f is the number of flavors, ψ_f are quarks of mass m_f in the fundamental representation, and μ is the quark chemical potential which is related to the baryon chemical potential μ_B as $\mu_B = 3\mu$. Properties of this theory have long been a subject of intense interest. (For a review, see [1].) Apart from its intrinsic theoretical appeal, this subject is important in astrophysics, especially in the study of neutron stars. The behavior of QCD at asymptotically large μ_B is well understood theoretically due to the asymptotic freedom, and QCD becomes a color superconductor as $\mu_B \to \infty$ [2].

At more phenomenologically realistic densities, QCD is strongly coupled, and thus not amenable to controlled analytic treatment. Although lattice Monte Carlo is very useful at $\mu_B = 0$, however, it runs into trouble at $\mu_B \neq 0$ due to the *fermion sign problem* – the fermion determinant $\prod_{f=1}^{N_f} \det \left(\gamma^\mu D_\mu + m_f + \mu\gamma^4 \right)$ becomes complex, rendering importance sampling exponentially difficult.

In order to circumvent this difficulty people have studied gauge theories which do not suffer from the sign problem at finite density. Consider QCD and QCD-like theories[1] of the form

$$\mathcal{L}_{\mathrm{G}} = \frac{1}{4g_{\mathrm{G}}^2} \operatorname{tr}(F_{\mu\nu}^{\mathrm{G}})^2 + \sum_{f=1}^{N_f} \bar{\psi}_f^{\mathrm{G}} \left(\gamma^\mu D_\mu^G + m_f + \mu_f\gamma^4 \right) \psi_f^{\mathrm{G}}, \tag{2}$$

where G is the gauge group e.g. $SU(N_c)$, μ_f is a generic quark chemical potential, and fermions ψ^{G} are not necessarily in the fundamental representation. The main examples are QCD with an isospin chemical potential μ_I (i.e. $N_f = 2$, $\mu_1 = -\mu_2 = \mu_I/2$) and degenerate mass $m_1 = m_2$ [3], two-color QCD of even number of flavors and degenerate mass [4], QCD with adjoint fermions [4], and $SO(2N_c)$ and $Sp(2N_c)$ Yang-Mills theories [5, 6]. However, while interesting, these theories have many qualitative differences from $N_c = 3$ QCD, such as *e.g.* explicitly broken flavor symmetry in the first case. Therefore it is important to understand *what we can learn from these theories*, or in other words, *in what sense they are similar to real QCD with the baryon chemical potential.*

In [5, 6], an answer to this question has been given. The statements are

- $SO(2N_c)$ YM with μ_B, $Sp(2N_c)$ YM with μ_B and $SU(N_c)$ QCD with μ_I are large-N_c equivalent both in the 't Hooft limit (N_f fixed) and the Veneziano limit (N_f/N_c fixed), everywhere in the T-μ plane. (Fermions are in the fundamental (vector) representation.)

- $SO(2N_c)$ YM with μ_B, $Sp(2N_c)$ YM with μ_B, $SU(N_c)$ QCD with μ_I and $SU(N_c)$ QCD with μ_B are large-N_c equivalent in the 't Hooft limit, outside the BEC/BCS crossover region of the former three theories. (Fermions are in the fundamental representation.)

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[1] In this paper we call $SU(N_c)$ Yang-Mills with N_f fundamental fermions 'QCD'. $SU(N_c)$ Yang-Mills with fermions in other representations and $SO(2N_c)/Sp(2N_c)$ theories are referred to 'QCD-like theories'.

- More generally, $SO(2N_c)$, $Sp(2N_c)$ and $SU(N_c)$ theories with fermion mass $m_1, \cdots, m_{N_f}$ and chemical potential $\mu_1, \cdots, \mu_{N_f}$ are equivalent. The signs of the quark chemical potentials can be flipped without spoiling the equivalence. (Fig. 2)

- $SO(2N_c)$ YM with the N_f complex adjoint fermions and μ_B, $SU(N_c)$ YM with the N_f complex adjoint fermions and μ_B, and $SU(N_c)$ YM with the $2N_f$ complex anti-symmetric fermions and μ_I are large-N_c equivalent everywhere in the T-μ plane.

These statements have been derived by using string-inspired large-N_c techniques [7, 8, 9, 10]. As shown in [5, 6], there are orbifold projections relating $SO(2N_c)$ and $Sp(2N_c)$ theories with μ_B, QCD with μ_B and QCD with μ_I (Fig. 1). At large-N_c, the orbifold equivalence guarantees these theories are equivalent in the sense a class of correlation functions (e.g. the expectation value of the chiral condensate and π^0 correlation functions) and the phase diagrams determined by such quantities coincide, as long as the projection symmetry is not broken spontaneously [10]. A similar argument shows QCD with adjoint fermions and μ_B is equivalent to QCD with fermions of two-index antisymmetric representation, which is a so-called the Corrigan-Ramond large-N_c limit, with μ_I (Fig. 3). In order for these equivalence to hold, orbifolding symmetries must not be broken spontaneously. This requirement is always satisfied for the equivalences between $SO(2N_c)$ YM with μ_B, $Sp(2N_c)$ YM with μ_B and QCD with isospin chemical potential. For the equivalences between these three theories and QCD with the baryonic chemical potential, 'outside the BEC/BCS crossover region' is required for the symmetry realization. This region ('outside the BEC/BCS crossover region') is relevant for the search for the 'QCD critical point', which attract intense interest over the decade. Our answer to the problem is strikingly simple – by using sign-free theories one can answer to the question. In the case of the two-flavor theory, QCD with μ_I is nothing but the phase-quenched version of QCD with μ_B. Therefore, the sign problem is merely an illusion, up to the $1/N_c$ correction. Furthermore, for gluonic observables, the leading $1/N_c$ corrections to the large-N_c limit which carries the information of the chemical potential are the same in these theories. We also argue that it might be possible to study the large-μ_B region, where the equivalence fails because of the symmetry breakdown, by adding certain deformations to the $SO(2N_c)$ theory while leaving the QCD with μ_B untouched.

This paper is organized as follows. In § 2 we show the absence of the sign in $SO(2N_c)$ theory (and also in $Sp(2N_c)$ theory when the mass and chemical potential are degenerated). In § 3, we review the orbifold equivalence and provide a proof to all order in perturbation[2]. As discussed in [10], projection symmetries must be unbroken in order for the equivalences to hold. In order to see the fate of the symmetries, we discuss the phase diagrams of $SO(2N_c)$, $Sp(2N_c)$ and $SU(N_c)$ theories in § 3.2.3. In § 4 we generalize the equivalences to analytically solvable toy models and confirm the equivalence explicitly. Results in this section strongly suggests the validity of the equivalences at nonperturbative level. § 5 is devoted for the conclusion and outlook.

This paper has been prepared for Seitaro Nakamura prize competition, and is based on papers [5] and [6]. We also provide a few new results.

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[2] We do not show the proof the equivalence between theories with adjoint and antisymmetric fermions. The proof for the theories with fundamental fermion can easily be modified for this case. The projection is the same as the one used in [11]. The equivalence between two $SU(N_c)$ theories (when the chemical potential is zero) has been found in [12].
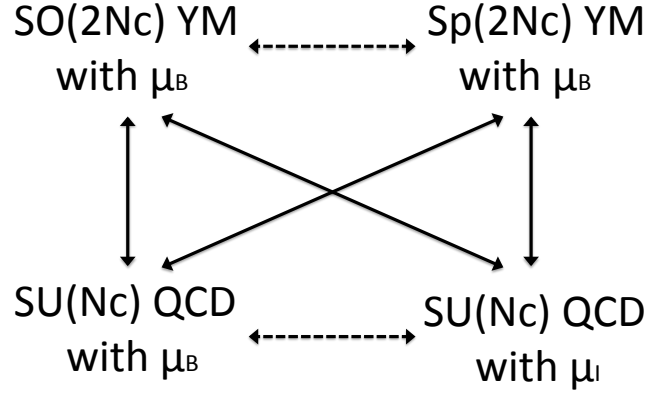
SO(2Nc) YM
with μ_B ⟵- - - - - -⟶ Sp(2Nc) YM
with μ_B

SU(Nc) QCD
with μ_B ⟵- - - - - -⟶ SU(Nc) QCD
with μ_I

Figure 1: A web of equivalences. Arrows with solid lines represents equivalences through orbifold projections. Arrows with dashed lines are the 'parent-parent' and 'daughter-daughter' equivalences which arise as combinations of the orbifold equivalences.
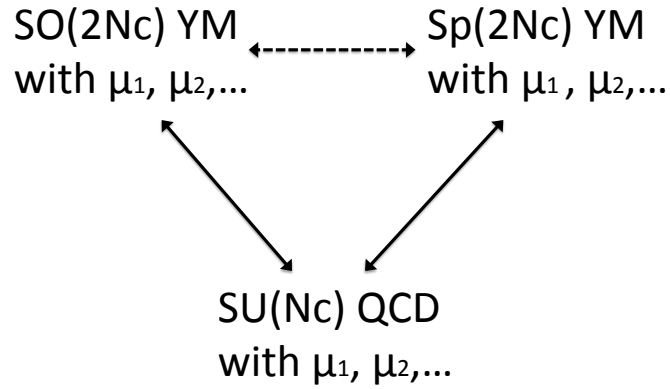
SO(2Nc) YM
with $\mu_1, \mu_2,...$ ⟵- - - - - -⟶ Sp(2Nc) YM
with $\mu_1, \mu_2,...$

SU(Nc) QCD
with $\mu_1, \mu_2,...$

Figure 2: More general version of the equivalences. Values of the quark chemical potentials can be different.
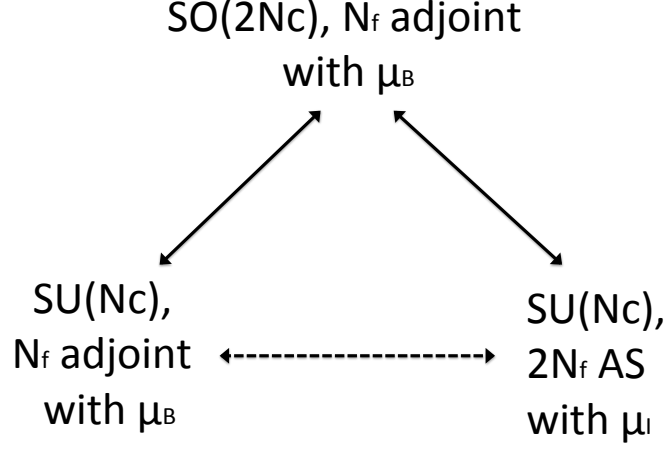
SU(Nc),
N_f adjoint
with μ_B

SU(Nc),
2N_f AS
with μ_I

Figure 3: Equivalences in the Corrigan-Ramond limit. $SU(N_c)$ YM with anti-symmetric fermions can be regarded as a special kind of large-N_c limit of three-color QCD (the Corrigan-Ramond limit), because anti-symmetric and fundamental representations are equivalent at $N_c = 3$. Unfortunately, $SU(N_c)$ YM with anti-symmetric fermions and μ_B cannot be incorporated in these equivalences.

2 Absence of the sign problem in $SO(2N_c)$ and $Sp(2N_c)$ theories

In this section we prove the absence of the sign problem in $SO(2N_c)$ and $Sp(2N_c)$ theories.

Let us start with $SU(N_c)$ theory. In four-dimensional Euclidean space, the gamma matrices can be taken to be Hermitian, $\gamma_\mu^\dagger = \gamma_\mu$. The covariant derivative $D_\mu^{\mathrm{SU}} = \partial_\mu + iA_\mu^{\mathrm{SU}}$, where A_μ^{SU} is an $N_c \times N_c$ Hermitian matrix, is skew-hermitian, $\left(D_\mu^{\mathrm{SU}}\right)^\dagger = -D_\mu^{\mathrm{SU}}$. Therefore, $\gamma_\mu D_\mu$ is also skew-Hermitian,

$$\left(\gamma^\mu D_\mu^{\mathrm{SU}}\right)^\dagger = -\gamma^\mu D_\mu^{\mathrm{SU}}. \tag{3}$$

Hence the eigenvalues of $\gamma^\mu D_\mu^{\mathrm{SU}}$ are pure imaginary. Furthermore the eigenvalues appear in pairs $\pm i\lambda$, where λ is real, because of the chiral symmetry $\gamma_5 \left(\gamma^\mu D_\mu^{\mathrm{SU}}\right) \gamma_5 = -\gamma^\mu D_\mu^{\mathrm{SU}}$. When the mass m_f is turned on, the eigenvalues are shifted to $\pm i\lambda + m_f$. Hence the eigenvalues appear with their complex conjugate (as long as m_f is real) and hence the determinant, which is the product of the eigenvalues, is real positive.

Once the chemical potential μ is turned on, this property is lost because

$$\left(\gamma^\mu D_\mu^{\mathrm{SU}} + \mu\gamma_4\right)^\dagger = -\gamma^\mu D_\mu^{\mathrm{SU}} + \mu\gamma_4, \tag{4}$$

and hence the determinant is complex in general. Note however that

$$\left[\det\left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)\right]^* = \det\left(\gamma^\mu D_\mu + m - \mu\gamma^4\right). \tag{5}$$

For this reason QCD with the isospin chemical potential and degenerate mass is sign-free; the determinant is

$$\det\left(\gamma^\mu D_\mu + m + \mu\gamma^4\right) \times \det\left(\gamma^\mu D_\mu + m - \mu\gamma^4\right) = \left|\det\left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)\right|^2 \geq 0. \tag{6}$$

In $SO(2N_c)$ theory, the sign is absent thanks to an additional anti-unitary symmetry. The crucial point is $D_\mu^{\rm SO}$ is real in the coordinate basis; both ∂_μ and $iA_\mu^{\rm SO}$ are real antisymmetric. Therefore

$$(C\gamma_5)\left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)(C\gamma_5)^{-1} = \left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)^*, \tag{7}$$

where C is the charge conjugation matrix satisfying $C\gamma_\mu C^{-1} = -\gamma_\mu^T = -\gamma_\mu^*$. It guarantees the pair structure of the eigenvalues (λ, λ^*); if v is an eigenvector with an eigenvalue λ, $(C\gamma_5)^{-1}v^*$ is another eigenvector with an eigenvalue λ^*, and furthermore, they are linearly independent even when λ is real [5].

In $Sp(2N_c)$ theory, a similar relation holds,

$$(J_cC\gamma_5)\left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)(J_cC\gamma_5)^{-1} = \left(\gamma^\mu D_\mu + m + \mu\gamma^4\right)^*, \tag{8}$$

where $J_c = i\sigma_2 \otimes 1_{N_c}$ (see § 3.1.2). Unlikely to the case of $SO(2N_c)$, this relation does not give the pair structure when λ is real, and hence only the reality, not the positivity, is guaranteed. However as long as the mass and chemical potential are degenerate ($m_1 = m_2$, $m_3 = m_4$, $\cdots$, $\mu_1 = \mu_2$, $\mu_3 = \mu_4$, $\cdots$) the determinant is real and positive.

3 Orbifold equivalence

3.1 Pure Yang-Mills theory

3.1.1 $SO(2N_c)$ to $SU(N_c)$

The notion of the orbifold equivalence came from the string theory [7, 8]. Soon it has been proven by using only field theory techniques [9, 10], without referring to the string theory. As a simple example, let us consider the equivalence between $SO(2N_c)$ and $SU(N_c)$ pure Yang-Mills theories. To perform an orbifold projection, one identifies a discrete subgroup of the symmetry group of the 'parent' theory, which is the $SO(2N_c)$ theory in this case, and require the fields to be invariant under the discrete symmetry. This gives a 'daughter' theory, which is $SU(N_c)$ YM. The orbifold projection uses a $\mathbb{Z}_2$ subgroup of the $SO(2N_c)$ gauge symmetry.

Let us take $J_c \in SO(2N_c)$ to be $J_c = i\sigma_2 \otimes 1_{N_c}$, which generates a $\mathbb{Z}_4$ subgroup of $SO(2N_c)$. Here 1_N is an $N \times N$ identity matrix. We require the gauge field A_μ to be invariant under

$$A_\mu \to J_c A_\mu J_c^{-1}, \tag{9}$$

which generates a $\mathbb{Z}_2$ subgroup of $SO(2N_c)$. A_μ can be written in $N_c \times N_c$ blocks as

$$A_\mu = i\begin{pmatrix} A_\mu^A + B_\mu^A & C_\mu^A - D_\mu^S \\ C_\mu^A + D_\mu^S & A_\mu^A - B_\mu^A \end{pmatrix}, \tag{10}$$

6

where fields with an 'A' ('S') superscript are anti-symmetric (symmetric) matrices. Under the $\mathbb{Z}_2$ symmetry, A_μ^A, D_μ^S are even while B_μ^A, C_μ^A are odd, and hence the orbifold projection sets $B_\mu^A = C_\mu^A = 0$; the 'daughter' field is

$$A_\mu^{\text{proj}} = i \begin{pmatrix} A_\mu^A & -D_\mu^S \\ D_\mu^S & A_\mu^A \end{pmatrix}. \tag{11}$$

By using a unitary matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{N_c} & i1_{N_c} \\ 1_{N_c} & -i1_{N_c} \end{pmatrix}, \tag{12}$$

it can be rewritten as

$$P A_\mu^{\text{proj}} P^{-1} = \begin{pmatrix} -\mathcal{A}_\mu^T & 0 \\ 0 & \mathcal{A}_\mu \end{pmatrix}, \tag{13}$$

where $\mathcal{A}_\mu \equiv D_\mu^S + iA_\mu^A$ is a $U(N_c)$ gauge field. However, the difference between $U(N_c)$ and $SU(N_c)$ is a $1/N_c^2$ correction and can be neglected at large-N_c. [3] The gauge part of the action after the orbifold projection is thus simply

$$\mathcal{L}^{\text{gauge,proj}} = \frac{2}{4g_{SO}^2} \text{ Tr } \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \tag{14}$$

where $\mathcal{F}_{\mu\nu}$ is the $SU(N_c)$ field strength. Let us identify it with the Lagrangian of the daughter theory times two,

$$\mathcal{L}_{\text{SO}} \to 2\mathcal{L}_{\text{SU}}, \tag{15}$$

or equivalently let us take $g_{SU}^2 = g_{SO}^2$, where g_{SU} is the gauge coupling constant of the $SU(N_c)$ theory. This factor two is necessary in order for the ground state energies, which are proportional to the degrees of freedom, to match. Then expectation values of the gauge-invariant operators in parent $\mathcal{O}^{(p)}[A_\mu]$ agree with the expectation values of the daughter theory, which is obtained by replacing A_μ with $\mathcal{A}_\mu^{\text{proj}}$, $\mathcal{O}^{(d)}[\mathcal{A}_\mu] \equiv \mathcal{O}^{(p)}[A_\mu^{\text{proj}}]$.

As a pedagogical demonstration, consider a planar diagram in Fig. 4. In order to obtain the $SU(N_c)$ diagram, we insert the projector $\mathcal{P}$ defined by

$$A_\mu^{\text{proj}} = \mathcal{P}(A_\mu^{\text{SO}}) \equiv \frac{1}{4} \sum_{n=0}^{3} J_c^n A_\mu^{\text{SO}} J_c^{-n} = \frac{1}{2} \left(A_\mu^{\text{SO}} + J_c A_\mu^{\text{SO}} J_c^{-1} \right). \tag{16}$$

to each propagator in the $SO(2N_c)$ diagram. Then the only difference, if exists, comes from the contractions of color indices. This additional kinematic factor multiplied to the $SU(N_c)$ diagram is

$$\sum_{n_i=0,1} \left(\frac{1}{2} \right)^{N_P} \cdot \text{tr}(J^{-n_1} J^{n_4} J^{n_5}) \cdot \text{tr}(J^{-n_2} J^{-n_4} J^{n_6}) \cdot \text{tr}(J^{-n_3} J^{-n_5} J^{-n_6}) \cdot \text{tr}(J^{n_1} J^{n_2} J^{n_3}), \tag{17}$$

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[3] When one studies $U(N_c)$ theory, it is difficult to control the $U(1)$ part and the effect of the chemical potential can disappear [13] by a lattice artifact. In order to avoid that one should simulate $SU(N_c)$ theory on the lattice.
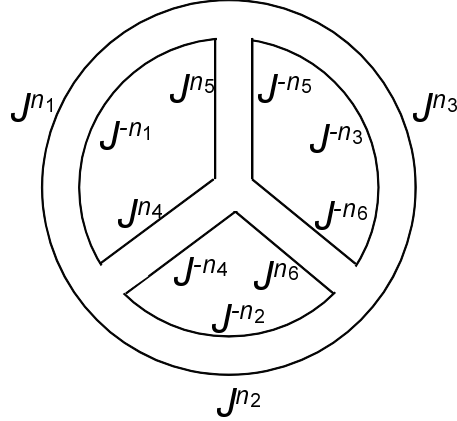
Figure 4: A vacuum planar diagram in the double-line notation.

where $J = -i\sigma_2$ is a 2×2 matrix and the factor $(1/2)^{N_P}$ comes from the projectors with $N_P = 6$, where N_P is the number of propagators. Because J satisfies a simple relation

$$TrJ^n = 0 \qquad \text{unless} \quad J^n = \pm\mathbf{1}_2, \tag{18}$$

it is nonvanishing only when

$$J^{-n_1}J^{n_4}J^{n_5} = \pm\mathbf{1}_2, \quad J^{-n_2}J^{-n_4}J^{n_6} = \pm\mathbf{1}_2, \quad J^{-n_3}J^{-n_5}J^{-n_6} = \pm\mathbf{1}_2, \quad J^{n_1}J^{n_2}J^{n_3} = \pm\mathbf{1}_2, \tag{19}$$

or equivalently

$$-n_1 + n_4 + n_5 = \text{even}, \quad -n_2 - n_4 + n_6 = \text{even}, \quad -n_3 - n_5 - n_6 = \text{even}, \quad n_1 + n_2 + n_3 = \text{even}. \tag{20}$$

Not all constraints are independent; actually the last one follows from the others, and hence, there are $N_L - 1 = 3$ independent constraints, where $N_L = 4$ is the number of index loops. By summing over n_i for all $N_P = 6$ propagators with $N_L - 1 = 3$ constraints, one obtains a factor of 2^{6-3}. Another factor 2^4 comes from the traces over color indices. Therefore, the total factor is

$$2^{-6} \cdot 2^{6-3} \cdot 2^4 = 2. \tag{21}$$

Generally, for given planar vacuum diagrams with N_P propagators and N_L loops, the projectors give a factor of $(1/2)^{N_P}$, the summation over n_i under the $N_L - 1$ constraints gives $2^{N_P-(N_L-1)}$, and the trace gives 2^{N_L}. The total factor is always 2:

$$2^{-N_P} \cdot 2^{N_P-(N_L-1)} \cdot 2^{N_L} = 2. \tag{22}$$

This factor 2 reflects the fact that the number of degrees of freedom in the parent theory is twice larger than that in the daughter theory. Hence the vacuum energy per degree of freedom is equivalent between these theories.

The counting does not apply for nonplanar diagrams. Indeed, one can easily check that the number of independent constraints is no longer $N_L - 1$, and the factor counted in (22) is generally

different from 2 [9]. This is why we need to take the large-N_c limit to suppress the nonplanar diagrams.

A few remarks are in order here. Firstly, in order for the orbifold equivalence to work, the projection symmetry must not be broken spontaneously. In this example, because the projection symmetry is embedded to the gauge transformation, it is not broken. In the following sections, we introduce matter fields and use flavor symmetries for the projection. Then the projection symmetry does break in a certain parameter region. Secondly, not all operators in two theories coincide. In the parent theory, only operators invariant under the projection symmetry is related to projected fields in the daughter. (In the above example this condition is not relevant because all the gauge invariant operators automatically satisfy this condition.) In the daughter, not all operators are obtained from the parent through the projection; in the present case, the projected operators are necessarily charge conjugation invariant.

3.1.2 $Sp(2N_c)$ to $SU(N_c)$

For $Sp(2N_c)$ gauge theory, the symplectic algebra $Sp(2N_c)$ formed by $2N_c \times 2N_c$ Hermitian matrices satisfying

$$J_c A^{\mathrm{Sp}} + (A^{\mathrm{Sp}})^T J_c = 0, \tag{23}$$

can be written using $N_c \times N_c$ matrices as

$$A_\mu^{\mathrm{Sp}} = \begin{pmatrix} iA_\mu^A + B_\mu^S & C_\mu^S - iD_\mu^S \\ C_\mu^S + iD_\mu^S & iA_\mu^A - B_\mu^S \end{pmatrix}, \tag{24}$$

where the fields A_μ^A (B_μ^S, C_μ^S, and D_μ^S) are anti-symmetric (symmetric) matrices. After the projection, one obtains $B_\mu^S = C_\mu^S = 0$. This gives $SU(N_c)$ gauge theory again.

3.2 Introducing fundamental fermions

3.2.1 Orbifold projection of fundamental fermions

In this section, we introduce the orbifold projection for fundamental fermions [5, 6].

Let us consider the effect of the orbifolding on ψ_f. By using $\omega = e^{i\pi/2} \in U(1)_B$, we define the projection by

$$\psi_f = \omega J_c \psi_f. \tag{25}$$

Writing

$$\begin{pmatrix} \psi_f^+ \\ \psi_f^- \end{pmatrix} = P\psi_f, \tag{26}$$

the action of the $\mathbb{Z}_2$ symmetry is just $(\psi_f^+, \psi_f^-) \to (-\psi_f^+, \psi_f^-)$. The projection consists of setting $\psi_a^+ = 0$.

The action of the daughter theory is (after rescaling the coupling constant)

$$\mathcal{L} = \frac{1}{4g_{SU}^2} \, \text{Tr} \, \mathcal{F}_{\mu\nu}^2 + \sum_{f=1}^{N_f} \bar{\psi}_f^{\text{SU}} \left(\gamma^\mu \mathcal{D}_\mu + m_q + \mu\gamma^4 \right) \psi_f^{\text{SU}}, \tag{27}$$

where $\mathcal{F}_{\mu\nu}$ is the field strength of the $SU(N_c)$ gauge field $\mathcal{A}_\mu = D_\mu^S + iA_\mu^A$, $\psi_a^{\text{SU}} = \psi_a^-$, and $\mathcal{D}_\mu = \partial_\mu + i\mathcal{A}_\mu$. This is an $SU(N_c)$ gauge theory with N_f flavors of fundamental Dirac fermions and the baryon chemical potential $\mu_B = \mu N_c$. So the orbifold projection relates $SO(2N_c)$ gauge theory to large N_c QCD.

On the other hand, in order to obtain fermions at finite μ_I for even N_f, we use $J_c \in SO(2N_c)$ [or $J_c \in Sp(2N_c)$] and $J_i \in SU(2)_{\text{isospin}}[\subset SU(N_f)]$ defined by

$$J_i = -i\sigma_2 \otimes 1_{N_f/2}. \tag{28}$$

We choose the projection condition to be

$$J_c \psi_{af}^{\text{SO(Sp)}} J_i^{-1} = \psi_{af}^{\text{SO(Sp)}}. \tag{29}$$

The flavor N_f-component fundamental fermion is decomposed into two $(N_f/2)$-component fields,

$$\psi^{\text{SO(Sp)}} = (\psi_i \ \psi_j), \tag{30}$$

with i and j being the isospin indices. If we define $\varphi_\pm = (\psi_\pm^i \mp i\psi_\pm^j)/\sqrt{2}$ and $\xi_\pm = (\psi_\pm^i \pm i\psi_\pm^j)/\sqrt{2}$, the fermions $\varphi_\pm$ survive but $\xi_\pm$ disappear after the projection (29). Because $\varphi_\pm$ couple to $(A_\mu^{\text{SU}})^C$ and A_μ^{SU} respectively, the action of the daughter theory is expressed as

$$\mathcal{L}_{\text{SU}} = \frac{1}{4g_{\text{SU}}^2} \, \text{tr}(F_{\mu\nu}^{\text{SU}})^2 + \sum_{f=1}^{N_f/2} \sum_\pm \bar{\psi}_{f\pm}^{\text{SU}} \left(\gamma^\mu D_\mu + m \pm \mu\gamma^4 \right) \psi_{f\pm}^{\text{SU}}, \tag{31}$$

where $\psi_+^{\text{SU}} = \sqrt{2}\varphi_-$ and $\psi_-^{\text{SU}} = \sqrt{2}\varphi_+^C$. This theory has the isospin chemical potential $\mu_I = 2\mu$.

3.2.2 't Hooft limit vs Veneziano limit

In the proof of the orbifold equivalence of the pure Yang-Mills theories shown in § 3.1, the conditions (20) are crucial. What happens when the fermions are introduced? Firstly note that two projections (25) and (29) are equivalent when the chemical potential is absent. Both are a $\mathbb{Z}_4$ subgroup of the flavor symmetry which mixes two Majorana flavors.

Once the chemical potential is turned on, they are not equivalent. The flavor symmetry J_i used in (29) satisfies the condition similar to (18). Therefore, the proof can be repeated straightforwardly; the only difference is some color-index loops are replaced with flavor-index loops. On the other hand, $\mathbb{Z}_4 \in U(1)_B$ used in (25) does not satisfy such a condition. Note however that one of the conditions in (20) is not independent and follows from other ones. Therefore, as long as the number of flavor-index loop is one, the proof holds. Because the fermion loops are suppressed by the factor N_f/N_c, the equivalence through (25) holds in the 't Hooft large-N_c limit (N_f fixed) while the one through (29) holds also in the Veneziano limit (N_f/N_c fixed).

The above argument has an implication for the $1/N_c$ correction. Let us consider QCD with μ_B and that with μ_I. In the 't Hooft large-N_c limit, gluonic operators trivially agree because the

fermions are not dynamical. Let us consider finite-N_c, say $N_c = 3$ and $N_f = 2$. Then the largest correction to the 't Hooft limit comes from one-fermion-loop planar diagrams, which, as we have seen, do not distinguish μ_B and μ_I. Therefore gluonic operators should behave similarly even quantitatively; the difference is at most $(N_f/N_c)^2$ (two-fermion-loop planar diagrams) or $(1/N_c^2) \cdot (N_f/N_c)$ (one-fermion-loop nonplanar diagrams). In particular, the deconfinement temperatures, which is determined from the Polyakov loop, should be close.

3.2.3 Symmetry realization and validity of the equivalence

As we have seen so far, $SU(N_c)$ QCD with μ_B/μ_I, $SO(2N_c)$ YM and $Sp(2N_c)$ YM should be equivalent in the large-N_c limit as long as the projection symmetries are not broken spontaneously. In this section we discuss the phase structures of these theories and clarify when the symmetries are broken. It turns out that $SU(N_c)$ QCD with μ_I, $SO(2N_c)$ YM and $Sp(2N_c)$ YM should be equivalent everywhere in $T - \mu$ parameter space. The equivalence to $SU(N_c)$ QCD with μ_B is not valid outside the BEC/BCS crossover region of other three theories. In § 3.2.4 we discuss a possible cure to this.

Let us start with $SO(2N_c)$ YM with μ_B. A crucial difference from QCD is that there is no distinction between 'matter' and 'antimatter' because the gauge group is real. In other words, 'fundamental' and 'antifundamental' representations are equivalent. For this reason, mesons in this theory are not necessarily neutral under $U(1)_B$; one can construct 'baryonic mesons' and 'antibaryonic mesons' out of two 'quarks' and 'antiquarks', respectively. Because they couple to μ_B, as we increase the value of μ_B the lightest 'baryonic meson' condenses at some point. Then the $U(1)_B$ symmetry is broken to $\mathbb{Z}_2$ and the equivalence to QCD with μ_B fails. (Note that we have used $\mathbb{Z}_4$ subgroup of $U(1)_B$ for the projection.)

In order to identify the lightest baryonic meson, let us consider the chiral symmetry breaking in this theory. When $m = \mu_B = 0$, the Lagrangian (2) has the $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_A$ symmetry at the classical level at first sight. However, chiral symmetry of the theory is known to be enhanced to $SU(2N_f)$. Here $U(1)_A$ is explicitly broken by the axial anomaly. One can actually rewrite the fermionic part of the Lagrangian (2) manifestly invariant under $SU(2N_f)$ using the new variable $\Psi = (\psi_L, \sigma_2 \psi_R^*)^T$:

$$\mathcal{L}_{\mathrm{f}} = i\Psi^\dagger \sigma_\mu D_\mu \Psi, \tag{32}$$

where $\sigma_\mu = (\sigma_k, -i)$ with the Pauli matrices σ_k ($k = 1, 2, 3$). The chiral symmetry $SU(2N_f)$ is spontaneously broken down to $SO(2N_f)$ by the formation of the chiral condensate $\langle \bar{\psi}\psi \rangle$, leading to the $2N_f^2 + N_f - 1$ Nambu-Goldstone bosons living on the coset space $SU(2N_f)/SO(2N_f)$: neutral pions $\Pi_a = \bar{\psi}\gamma_5 P_a \psi$, 'baryonic pions' (or simply 'diquark') $\Sigma_S = \psi^T C\gamma_5 Q_S \psi$ and 'antibaryonic pions' $\Sigma_S^\dagger = \psi^\dagger C\gamma_5 Q_S \psi^*$. It is easy to see the fate of these bosons under the orbifold projection. The projection to QCD with μ_B maps neutral pions to pions in QCD, and baryonic and antibaryonic pions are projected away. On the other hand, the projection to QCD with μ_I sends neutral/baryonic/antibaryonic pions to π^0, π^+ and π^-, respectively. Therefore the (baryonic) pions in $SO(2N_c)$ YM and those in QCD have the same mass m_π. In the same way as the π^+ condensation in QCD with μ_I at $\mu = m_\pi/2$, baryonic pions condense at $\mu = m_\pi/2$ (Fig. 5 and Fig. 6).

At sufficiently large μ, the one-gluon exchange interaction in the $\psi\psi$-channel is attractive in the color symmetric channel, leading to the condensation of the diquark pairing $\langle \psi^T C\gamma_5 Q_S \psi \rangle$. This
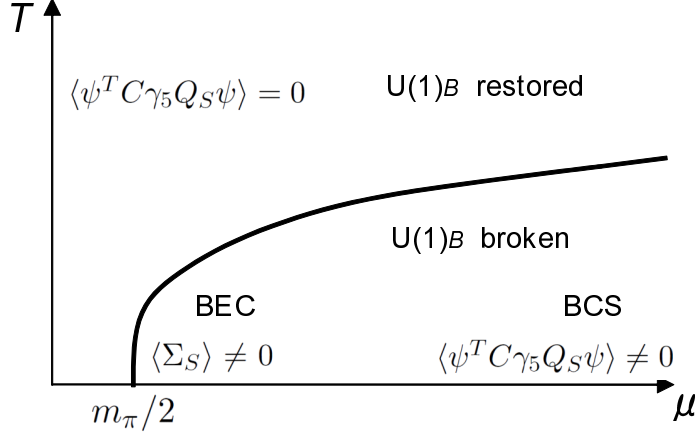
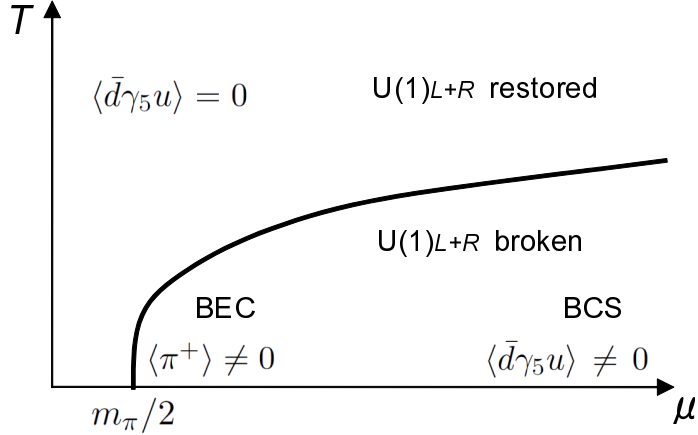Figure 5: Phase diagram of $SO(2N_c)$ gauge theory at finite μ_B. (Figure taken from [6].)

Figure 6: Phase diagram of QCD at finite $\mu_I = 2\mu$. (Figure taken from [6].)

diquark condensate does not break $SO(2N_c)$ symmetry. This BCS pairing has the same quantum numbers and breaks the same $U(1)_B$ symmetry as the BEC $\langle \Sigma_S \rangle$ at small μ_B, and there should be no phase transition for $\mu > m_\pi/2$ along μ axis. The phase diagram of this theory is similar to that of QCD at finite μ_I, as shown in Fig. 5 and Fig. 6. This is because the condensates in two theories are related each other through the orbifold projection, and furthermore, the condensation does not break the flavor symmetry used for the projection.

In the same manner, $Sp(2N_c)$ YM and QCD with μ_I are equivalent everywhere in $T - \mu$ plane; see Fig. 7. (For further details, see [6].)

QCD with μ_B behaves rather differently, because μ_B does not couple to mesons. This does not lead to a contradiction, however. Because baryons are much heavier than pions, phenomena characteristic to QCD with μ_B (e.g. formation of hadronic matter) takes place after the equivalence is gone due to the $U(1)_B$ breakdown in $SO(2N_c)$ and $Sp(2N_c)$ Yang-Mills.
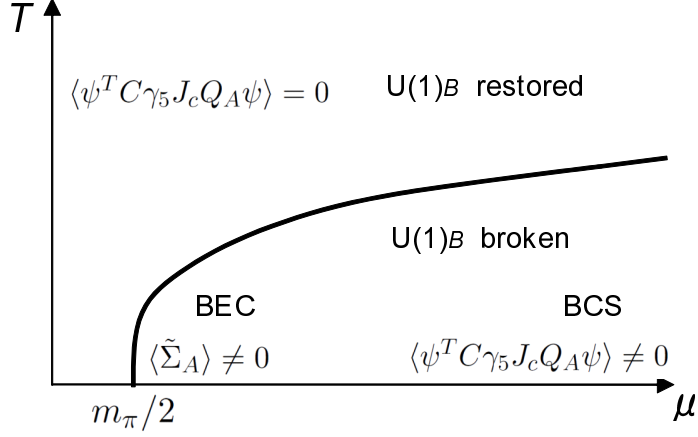
Figure 7: Phase diagram of $Sp(2N_c)$ gauge theory at finite μ_B. $\tilde{\Sigma}_A = \psi^T C \gamma_5 J_c Q_A \psi$, where Q_A ($A = 1, 2, \cdots, N_f(N_f - 1)/2$) are antisymmetric $N_f \times N_f$ matrices in the flavor space. (Figure taken from [6].)

3.2.4   Breakdown of the equivalence and its cure

As discussed in [10], the symmetries used in the orbifold projection must not be spontaneously broken in order for an orbifold equivalence to hold *non-perturbatively*. In our case, the $\mathbb{Z}_4 \in U(1)_B$ of $SO(2N_c)$ and $Sp(2N_c)$ YM, which is necessary to relate them to QCD with μ_B, breaks to $\mathbb{Z}_2$ when $\mu > m_\pi/2$ due to 'baryonic pion' condensation.

In order to recover the equivalence, one can add a deformation term to the $SO(2N_c)$ theory to prevent baryonic pion condensation and protect $U(1)_B$. A natural choice for a deformation term is

$$\mathcal{L}_{SO}^d = \frac{c^2}{\Lambda^2} \sum_{a,b} S_{ab}^\dagger S_{ab} \tag{33}$$

where $\Lambda \sim \Lambda_{QCD}$. The deformed theory should be viewed as an effective field theory, defined with an implicit (lattice) cutoff, since the deformation is an irrelevant operator. At large-N_c, the expectation value of this term factorizes to $\langle S_{ab} \rangle$ and its complex conjugate, and hence the baryonic pion condensation costs $\mathcal{O}(c^2)$ of energy. Therefore, provided c is large enough, the deformation prevents baryonic pion condensation, saving the validity of the orbifold equivalence between the *deformed $SO(2N_c)$* gauge theory and large N_c QCD.

Once μ becomes as large as Λ_{QCD}, other baryonic mesons might condense, breaking $U(1)_B$. Whether this actually happens can only be answered by lattice simulations. If some baryonic mesons do condense, one can introduce deformations (*e.g.*, of the form $P_{ab}^\dagger P^{ab}$ where $P_{ab} = q_a^T C q_b$ for the pseudoscalar baryonic mesons) to prevent it, just as was done for baryonic pions above. Note that the deformation terms do not survive the orbifold projection to QCD. Thus the value of c does not affect the correlation functions of neutral operators so long as it is large enough to prevent baryonic pion condensation.

In order for the numerical simulation, one has to 'integrate in' the auxiliary fields so that the action becomes bilinear in the fermion. Whether or not the auxiliary fields introduce the sign

problem is not an easy question. In [5] it has been shown that a similar deformation can be introduced without spoiling the sign-free nature. For more on the deformation, see [14].

4 Demonstration of the nonperturbative equivalence in solvable models

If the orbifold equivalence holds between the gauge theories, it is natural to expect that the equivalence should hold also in solvable toy models which are believed to capture essential features of the gauge theories. In this section, we consider the chiral random matrix theory (RMT) [15], and the holographic D3/D7 model which can be studied via AdS/CFT correspondence. We show that the perturbative proof applies also in these cases, and furthermore, we confirm the equivalence at nonperturbative level. These results strongly suggest that the orbifold equivalence of the gauge theories hold nonperturbatively.

4.1 Chiral random matrix theories

The partition function of the RMT is given by an integral over a Gaussian random matrix ensemble, instead of the average over the gauge field of the original Yang-Mills action:

$$
Z = \int d\Phi \prod_{i=1}^{N_f} \det \mathcal{D} \ e^{-\frac{N\beta}{2} G^2 \operatorname{tr} \Phi^\dagger \Phi},
\tag{34}
$$

where Φ is an $N \times (N+\nu)$ random matrix element, N is the size of the system, ν is the topological charge, and the Dirac operator $\mathcal{D}$ with quark mass m_f is given by

$$
\mathcal{D} = \left(
\begin{array}{cc}
m_f \mathbf{1} & \Phi + \mu \mathbf{1} \\
-\Phi^\dagger + \mu \mathbf{1} & m_f \mathbf{1}
\end{array}
\right).
\tag{35}
$$

We also introduced a suitable normalization with the parameter G in the Gaussian. Note that there is no spacetime coordinate in the theory; the size of the matrix N corresponds to the spacetime volume. It is taken to infinity in the end, corresponding to the thermodynamic limit.

Depending on the anti-unitary symmetries of the Dirac operator, Φ is chosen as the real, complex, or quaternion real [see (40) for the definition] matrices denoted by the Dyson index $\beta = 1$, $\beta = 2$, and $\beta = 4$, respectively. The value of β corresponds to the degrees of freedom per matrix element. QCD and QCD-like theories corresponding to each universality class are [19]:

- $\beta = 1$: two-color QCD and $Sp(2N_c)$ gauge theory.

- $\beta = 2$: $SU(N_c)$ QCD ($N_c \geq 3$).

- $\beta = 4$: $SU(N_c)$ QCD with adjoint fermions and $SO(N_c)$ gauge theory.

The effect of temperature T can be incorporated as the (first) Matsubara frequencies by changing $\mu \to \mu + iT$ for one half of the determinant and $\mu \to \mu - iT$ for the other half of the determinant in the simplest model [18].

The RMT is exactly equivalent to QCD in the ϵ-regime [20]

$$\frac{1}{\Lambda_\chi} \ll L \ll \frac{1}{m_\pi}, \qquad \mu L \ll 1, \tag{36}$$

where L is the typical scale of the system, because QCD reduces to a theory of zero momentum modes of pions. In this regime, the system has a universality, i.e., the dynamics depends only on the symmetry breaking pattern and is independent of the microscopic details; QCD can be replaced by the RMT with the same global symmetry breaking pattern. Outside the ϵ-regime, the universality is lost. However, the RMT is still useful as a schematic model to study the qualitative properties of QCD such as the phase structure at finite T and μ [18]. The advantage of the RMT is that it can be solved analytically although QCD itself cannot be.

As mentioned above, the RMT has the size of the matrix N, which should be taken to infinite (thermodynamic limit) in the end. In this sense, the RMT is a "large-N" matrix model, and hence, the perturbative proof of the orbifold equivalence given in Sec. 3 is directly applicable. Note that the size of the random matrix is not related to the number of color N_c. The RMT is analytically solvable and hence the nonperturbative orbifold equivalence can be checked explicitly.

4.1.1 Orbifold projections in the chiral random matrix theories

In this section, we construct the orbifold projections in the chiral random matrix theories (RMTs) between $\beta = 4$, $\beta = 2$, and $\beta = 1$. Thereby a class of observables in the RMTs with the different Dyson indices are found to be identical to each other. In the following, we will concentrate on the RMT at finite μ and $T = 0$, which can be easily generalized to nonzero T. For simplicity, we set $\nu = 0$ and quark masses to be common, $m_f = m$.

The relationships between these RMTs via orbifold projections are summarized in Fig. 8. We start with the $\beta = 4$ or $\beta = 1$ RMT at finite μ_B with the size of Φ being $2N$, and define the orbifold projection to the $\beta = 2$ RMT at finite μ_B or μ_I with the size N.

The action of the $\beta = 4$ RMT is given by

$$Z = \int d\Phi d\Psi \ e^{-S}, \qquad S = S_B + S_F, \tag{37}$$

where

$$S_B = \frac{N\beta}{2} G^2 \operatorname{tr} \Phi^\dagger \Phi, \qquad S_F = \sum_{f=1}^{N_f} \bar{\Psi}_f \mathcal{D} \Psi_f, \tag{38}$$

and

$$\mathcal{D} = \begin{pmatrix} m\mathbf{1}_{2N} & \Phi + \mu\mathbf{1}_{2N} \\ -\Phi^\dagger + \mu\mathbf{1}_{2N} & m\mathbf{1}_{2N} \end{pmatrix}. \tag{39}$$

Here Ψ_f are complex Grassmann $4N$-component vectors and Φ is a $2N \times 2N$ quaternion real matrix of the form:

$$\Phi \equiv \sum_{\mu=0}^{3} a^\mu i\sigma_\mu = \begin{pmatrix} a^0 + ia^3 & a^2 + ia^1 \\ -a^2 + ia^1 & a^0 - ia^3 \end{pmatrix}, \tag{40}$$
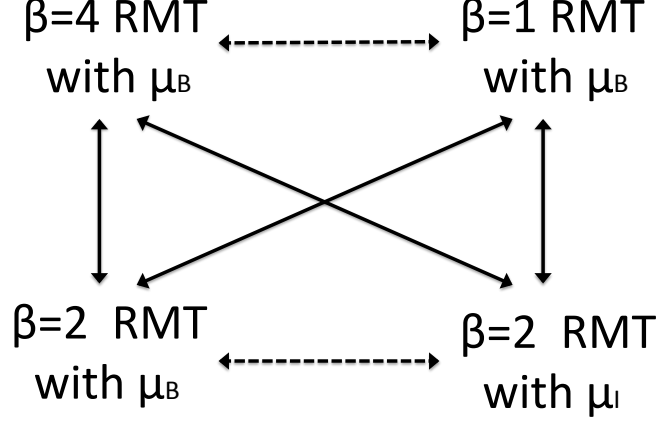
15

Figure 8: Relations between $\beta = 2$ RMT at finite μ_B or μ_I and $\beta = 4$ and $\beta = 1$ RMTs at finite μ_B through orbifolding. $\beta = 2$ RMT at small and large μ_I can be obtained from $\beta = 4$ and $\beta = 1$ RMTs at small and large μ_B. $\beta = 2$ RMT at small μ_B can also be obtained from $\beta = 4$ RMT at small μ_B, while $\beta = 2$ RMT at large μ_B inside the BEC-BCS crossover region cannot.

where a^μ are $N \times N$ real matrices and $\sigma_\mu = (-i, \sigma_k)$ with Pauli matrices σ_k.

For the bosonic matrix Φ, we impose the projection condition

$$J_c \Phi J_c^{-1} = \Phi, \qquad J_c \equiv \begin{pmatrix} & -\mathbf{1}_N \\ \mathbf{1}_N & \end{pmatrix}. \tag{41}$$

Then we obtain

$$\Phi^{\text{proj}} = \begin{pmatrix} a^0 & a^2 \\ -a^2 & a^0 \end{pmatrix}, \tag{42}$$

which is equivalent to two copies of a $N \times N$ complex matrix after a unitary transformation

$$P \Phi^{\text{proj}} P^{-1} = \begin{pmatrix} \phi^* & 0 \\ 0 & \phi \end{pmatrix} \equiv \Phi_{\beta=2}, \qquad P \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_N & i\mathbf{1}_N \\ \mathbf{1}_N & -i\mathbf{1}_N \end{pmatrix}, \tag{43}$$

where $\phi = a^0 + ia^2$. The bosonic part of the action is mapped to the one for the $\beta = 2$ RMT. Note that the factor 2 in (15) is reflected in the difference of normalization between $\beta = 4$ and $\beta = 2$ in (38) if the trace for $\beta = 4$ is defined as the so-call "QTr" which is one-half the usual trace.

In order to define a projection for the fermions, we write Ψ by using two $2N$-component fermions ψ_R and ψ_L as

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \tag{44}$$

Here ψ_R and ψ_L are further decomposed into two N-component fermions

$$\psi_R = \begin{pmatrix} \psi_R^1 \\ \psi_R^2 \end{pmatrix}, \qquad \psi_L = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}, \tag{45}$$

where the flavor index is suppressed for simplicity. Then it is straightforward to check that the projection

$$\psi_R = \omega J_c \psi_R, \qquad \psi_L = \omega J_c \psi_L, \tag{46}$$

where $\omega = e^{i\pi/2}$ as before, provides us with the $\beta = 2$ RMT at finite baryon chemical potential. In a similar manner, the $\beta = 2$ RMT at finite isospin chemical potential can be obtained by imposing the projection condition

$$J_c \psi_R J_i^{-1} = \psi_R, \qquad J_c \psi_L J_i^{-1} = \psi_L. \tag{47}$$

The $\beta = 2$ RMT can also be obtained from the $\beta = 1$ RMT. We start with the action of the $\beta = 1$ RMT given by (38), but this time Φ is a $2N \times 2N$ real matrix, which can be parametrized as

$$\Phi = \begin{pmatrix} a^0 + a^3 & a^2 + a^1 \\ -a^2 + a^1 & a^0 - a^3 \end{pmatrix}, \tag{48}$$

where a^μ are $N \times N$ real matrices. Note that the only change in this expression compared with (40) is that the factors i in front of a^0 and a^3 are absent. Then one can easily find that the same projection conditions for Φ and Ψ in the previous subsection gives the $\beta = 2$ RMT at finite μ_B or finite μ_I.

4.1.2 Explicit demonstration of the nonperturbative equivalence

The orbifold equivalence in the RMT predicts that the $\beta = 4$ and $\beta = 1$ RMTs at finite μ_B and $\beta = 2$ RMT at finite μ_I are equivalent to each other for any m, T and μ in the neutral sector, to all order in perturbation theory. Outside the (baryonic) pion condensation phase, the above three theories must also be equivalent to the $\beta = 2$ RMT at finite μ_B. Below we verify the nonperturbative equivalence by computing the effective potential of each RMT explicitly.

For the $\beta = 2$ and $\beta = 1$ RMTs, the effective potentials are computed in [16, 17], and the coincidence of the effective potential of the $\beta = 2$ RMT at finite μ_B (at $\mu_I = 0$) and that of the $\beta = 2$ RMT at finite μ_I (at $\mu_B = 0$) outside the pion condensed phase has been pointed out. Below we first summarize the effective potential of the $N_f = 2$ RMT with the quark mass m_f and the chemical potential μ_f for each flavor, $f = 1, 2$. The average baryon and isospin chemical potentials $\bar{\mu}_B$ and $\bar{\mu}_I$ are defined as

$$\bar{\mu}_B \equiv \frac{\mu_B}{N_c} = \frac{1}{2}(\mu_1 + \mu_2), \tag{49}$$

$$\bar{\mu}_I \equiv \frac{\mu_I}{2} = \frac{1}{2}(\mu_1 - \mu_2). \tag{50}$$

We denote the parameters of the RMT representing the chiral condensate σ_f, pion condensate ρ, and diquark (baryonic pion) condensate Δ, and their sources as m_f, λ, and J, respectively. Then

we show that the equivalence holds nonperturbatively between RMTs as a natural consequence of the orbifold projections. The importance of the unbroken projection symmetry will become clear through the argument.

Effective potential of $\beta = 4$

We first consider the $\beta = 4$ RMT with degenerate quark mass $m_f = m$ at finite baryon chemical potential $\mu_f = \bar{\mu}_B$. The effective potential is [6]

$$\Omega_{\beta=4} = 16G^2 \left[\left(\sigma - \frac{m}{2} \right)^2 + \left(\Delta - \frac{J}{2} \right)^2 \right] - 2 \sum_{\pm} \ln[4\sigma^2 + 4\Delta^2 - (\bar{\mu}_B \pm iT)^2]. \tag{51}$$

The chiral condensate and the diquark condensate are expressed using σ and Δ as

$$\langle \bar{u}u \rangle_{\beta=4} = \frac{1}{4N} \partial_m \ln Z_{\beta=4} \Big|_{m=0} = -4G^2 \sigma_{\beta=4}, \tag{52}$$

$$\langle \bar{u}^T C\gamma_5 u \rangle_{\beta=4} = \frac{1}{4N} \partial_J \ln Z_{\beta=4} \Big|_{J=0} = -4G^2 \Delta_{\beta=4}. \tag{53}$$

Effective potential of $\beta = 2$

The effective potential of $\beta = 2$ is [16]

$$\begin{aligned}
\Omega_{\beta=2} &= G^2[(\sigma_1 - m_1)^2 + (\sigma_2 - m_2)^2 + 2(\rho - \lambda)^2] \\
&\quad - \frac{1}{2} \sum_{\pm} \ln[(\sigma_1 + \mu_1 \pm iT)(\sigma_2 - \mu_2 \mp iT) + \rho^2][(\sigma_1 - \mu_1 \mp iT)(\sigma_2 + \mu_2 \pm iT) + \rho^2].
\end{aligned} \tag{54}$$

The chiral condensate and pion condensate are

$$\langle \bar{u}u \rangle_{\beta=2} = \frac{1}{2N} \partial_{m_1} \ln Z_{\beta=2} \Big|_{m_1=0} = -G^2 \sigma_{\beta=2}, \tag{55}$$

$$\langle \bar{d}\gamma^5 u \rangle_{\beta=2} = \frac{1}{4N} \partial_\lambda \ln Z_{\beta=2} \Big|_{\lambda=0} = -G^2 \rho_{\beta=2}. \tag{56}$$

Note that, as long as $\rho = 0$ (i.e., outside the pion condensed phase), the potential (54) is a function of $\mu_1^2 = (\bar{\mu}_B + \bar{\mu}_I)^2$ and $\mu_2^2 = (\bar{\mu}_B - \bar{\mu}_I)^2$. Therefore,

$$\Omega_{\beta=2}(\bar{\mu}_B = \mu, \bar{\mu}_I = 0) = \Omega_{\beta=2}(\bar{\mu}_I = \mu, \bar{\mu}_B = 0) \quad \text{for} \quad \rho = 0. \tag{57}$$

Here $\rho = 0$ means that the projection symmetry, which is used for the orbifolding in Sec. 4.1.1, is not broken. From (57), the magnitude of the chiral condensate σ and the critical temperature of chiral phase transition T^σ in each theory coincide,

$$\sigma_{\beta=2}(\bar{\mu}_B)|_{\bar{\mu}_I=0} = \sigma_{\beta=2}(\bar{\mu}_I)|_{\bar{\mu}_B=0} \quad \text{for} \quad \rho = 0, \tag{58}$$

$$T^\sigma_{\beta=2}(\bar{\mu}_B)|_{\bar{\mu}_I=0} = T^\sigma_{\beta=2}(\bar{\mu}_I)|_{\bar{\mu}_B=0} \quad \text{for} \quad \rho = 0, \tag{59}$$

as a consequence of the orbifold equivalence. Especially, this shows that the phase-quenched approximation for $\sigma_{\beta=2}(\bar{\mu}_B)$ and $T^\sigma_{\beta=2}(\bar{\mu}_B)$ works outside the pion condensed phase, as mentioned in [16].

Note that, even though the effective potentials are identical for $\rho = 0$ in (57), the partition functions themselves are not generally the same. This is because the pre-exponential factor also contributes to the partition function, which is not taken into account in computing the effective potential. Therefore, the sign problem measured as the phase of the partition function can be severe inside as well as outside the pion condensed phase [21]. The result here shows that the phase-quenched approximation is exact for the observables above independently of the severity of the sign problem, as long as $\rho = 0$.

Effective potential of $\beta = 1$

The effective potential of $\beta = 1$ is [17]

$$
\begin{aligned}
\Omega_{\beta=1} &= G^2[(\sigma_1 - m_1)^2 + (\sigma_2 - m_2)^2 + 2(\rho - \lambda)^2 + 2(\Delta - J)^2] \\
&\quad - \frac{1}{4}\sum_{\pm} \ln\{[(\sigma_1 + \mu_1 \pm iT)(\sigma_2 - \mu_2 \mp iT) + \rho^2 + \Delta^2] \\
&\qquad \times [(\sigma_1 - \mu_1 \pm iT)(\sigma_2 + \mu_2 \mp iT) + \rho^2 + \Delta^2] + 4\Delta^2 \mu_1 \mu_2\} \\
&\qquad \times \{[(\sigma_1 - \mu_1 \mp iT)(\sigma_2 + \mu_2 \pm iT) + \rho^2 + \Delta^2] \\
&\qquad \times (\sigma_1 + \mu_1 \mp iT)(\sigma_2 - \mu_2 \pm iT) + \rho^2 + \Delta^2] + 4\Delta^2 \mu_1 \mu_2\}.
\end{aligned}
\tag{60}
$$

The chiral condensate, pion condensate, and diquark condensate are

$$
\langle \bar{u}u \rangle_{\beta=1} = \frac{1}{2N}\partial_{m_1} \ln Z_{\beta=1}\Big|_{m_1=0} = -G^2 \sigma_{\beta=1},
\tag{61}
$$

$$
\langle \bar{d}\gamma^5 u \rangle_{\beta=1} = \frac{1}{4N}\partial_{\lambda} \ln Z_{\beta=1}\Big|_{\lambda=0} = -G^2 \rho_{\beta=1}.
\tag{62}
$$

$$
\langle d^T C \gamma_5 u \rangle_{\beta=1} = \frac{1}{4N}\partial_{J} \ln Z_{\beta=1}\Big|_{J=0} = -G^2 \Delta_{\beta=1}.
\tag{63}
$$

The potential (60) has the symmetry

$$
\Omega_{\beta=1}(\Delta, \rho, \mu_1, \mu_2) = \Omega_{\beta=1}(\rho, -\Delta, \mu_1, -\mu_2),
\tag{64}
$$

due to the $\bar{\mu}_B \leftrightarrow \bar{\mu}_I$ symmetry for $\beta = 1$. Note that this symmetry has nothing to do with the orbifold equivalence.

Nonperturbative orbifold equivalence between $\beta = 4$, $\beta = 2$, and $\beta = 1$

By comparing (51), (54), and (60), and by using the $\bar{\mu}_B \leftrightarrow \bar{\mu}_I$ symmetry for $\beta = 1$, one finds the relation (note that $\Delta = 0$ at $\bar{\mu}_B = 0$ and $\rho = 0$ at $\bar{\mu}_I = 0$):

$$
\begin{aligned}
\Omega_{\beta=4}(2\sigma_{\beta=4}, 2\Delta_{\beta=4})|_{\bar{\mu}_B=\mu, \bar{\mu}_I=0} &= 2\Omega_{\beta=2}(\sigma_{\beta=2}, \rho_{\beta=2})|_{\bar{\mu}_I=\mu, \bar{\mu}_B=0} \\
&= 2\Omega_{\beta=1}(\sigma_{\beta=1}, \Delta_{\beta=1})|_{\bar{\mu}_B=\mu, \bar{\mu}_I=0}.
\end{aligned}
\tag{65}
$$

Unlike the relation (57), this is valid not only for $\rho = 0$ (or $\Delta = 0$) but also for $\rho \neq 0$ (or $\Delta \neq 0$). This is expected because the condensation does not break the projection symmetry, as discussed

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in § 3. The factor 2 of the effective potentials, mentioned in (15), comes from the fact that the $\beta = 4$ RMT with the size of Φ being $2N$ twice more degrees of freedom compared to the $\beta = 2$ or $\beta = 1$ RMT with the size N. The origin of the factor 2 for the arguments of the potential of the $\beta = 4$ RMT is the same; fermions in $\beta = 4$ theory has twice larger degrees of freedom, $\langle \bar{u}u \rangle_{\beta=4} = 2\langle \bar{u}u \rangle_{\beta=2} = 2\langle \bar{u}u \rangle_{\beta=1}$. The relation (65) leads to the coincidence of the magnitudes of the order parameters (up to the factor 2) and the critical temperatures,

$$2\sigma_{\beta=4}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0} = \sigma_{\beta=2}(\bar{\mu}_I = \mu)|_{\bar{\mu}_B=0} = \sigma_{\beta=1}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0}, \qquad (66)$$

$$2\Delta_{\beta=4}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0} = \rho_{\beta=2}(\bar{\mu}_I = \mu)|_{\bar{\mu}_B=0} = \Delta_{\beta=1}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0}, \qquad (67)$$

$$T^{\sigma}_{\beta=4}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0} = T^{\sigma}_{\beta=2}(\bar{\mu}_I = \mu)|_{\bar{\mu}_B=0} = T^{\sigma}_{\beta=1}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0}, \qquad (68)$$

$$T^{\Delta}_{\beta=4}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0} = T^{\rho}_{\beta=2}(\bar{\mu}_I = \mu)|_{\bar{\mu}_B=0} = T^{\Delta}_{\beta=1}(\bar{\mu}_B = \mu)|_{\bar{\mu}_I=0}, \qquad (69)$$

which are expected as a consequence of the orbifold equivalence. We note that, the equivalence of the neutral order parameters and the critical temperatures should be satisfied in the original QCD and QCD-like theories as we claimed in §3, while the effective potentials will not necessarily coincide in QCD. The RMT has much less degrees of freedom; the effective potential is a function of only the neutral order parameters, and furthermore, all the moments are identical due to the orbifold equivalence. As a result, the effective potentials must be identical. In QCD and QCD-like theories, the effective potentials depend also on non-neutral observables, and hence the effective potentials are not identical in general.

4.2 Holographic models

Among other interesting toy models are supersymmetric analogues of large-N_c QCD which have gravity dual description. In [22] the D3/D7 system with chemical potential has been studied (Fig. 9). The starting point is 4d $\mathcal{N} = 4$ $U(2N_c)$ super Yang-Mills, which is dual to type IIB supergravity in $AdS_5 \times S^5$. We introduce N_f D7 branes winding on three-cycle of S^5. Then open strings stretching between D3 and D7 can be regarded as 'quarks' with 'flavor symmetry' $U(N_f)$. In the 't Hooft limit $N_f/N_c \to 0$, D7-branes behave as probes, and their dynamics is described by the Dirac-Born-Infeld (DBI) action on the $AdS_5 \times S^5$ background. The isospin chemical potential μ_I can be introduced through the boundary condition of the gauge field in the DBI action. This theory can be projected to an $SO(2N_c)$ theory with μ_B through an orientifold projection, and further down to $U(N_c)$ theory with μ_B. However because the orientifold projection does not change the local structure of the brane configuration, the equation of motion remains untouched as long as the projection symmetry is not broken spontaneously. Therefore, the dynamics of mesons determined by the DBI action coincide, and the orbifold equivalence follows. Note that this is a 'nonperturbative proof', if we assume the AdS/CFT duality holds nonperturbatively.

Actually these theories are solved in [23] ($U(N_c)$ with μ_B) and [24, 25] ($U(N_c)$ with μ_I), and as long as the necessary symmetry is intact the equivalence can be seen explicitly.

5 Conclusion and outlook

We have pointed out that QCD and various QCD-like theories with chemical potentials are equivalent at large-N_c through the orbifold equivalence, at least to all order in perturbation theory. QCD with the isospin chemical potential and $SO(2N_c)/Sp(2N_c)$ Yang-Mills with the
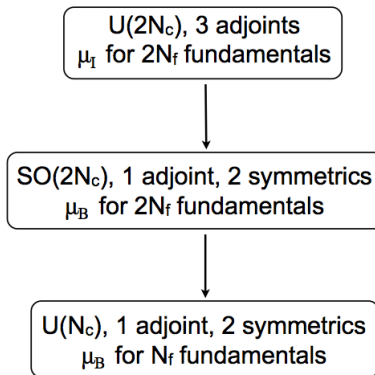
Figure 9: $\mathcal{N} = 1$ Supersymmetric version of the orbifold projection which has a holographic realization.

baryon chemical potential are equivalent everywhere in the T-μ plane, and furthermore, they are equivalent to QCD with the baryon chemical potential outside the BEC-BCS crossover region. We suggested the equivalence to QCD with the baryon chemical potential may extend inside the crossover region by deforming the other theories so that the transition to BEC-BCS phase is prevented, while keeping the sign-free nature.

Our result has immediate implication for the study of the chiral and deconfinement transitions in high-T, small-μ region. In this region it is reasonable to assume the N_f/N_c expansion is not bad and hence we can expect that the Monte-Carlo results of the QCD with isospin chemical potential describe the QCD with the baryon chemical potential with rather good accuracy. (Note that the deviation is only $O((N_f/N_c)^2)$, as explained in § 3.2.2.) Furthermore, by using the $SO(2N_c)$ theory, one can study three-flavor theory without suffering from the sign problem. In a similar manner, from simulation results of the two-color QCD and adjoint QCD, which belong to the same universality classes as $Sp(2N_c)$ and $SO(2N_c)$ theories, respectively, one can extract information relevant for $SU(3)$ QCD with μ_B. [4] Similar study e.g. phase quenched simulation of $SU(3)$ 3-flavor QCD has been performed [26] and the result suggests the QCD critical point does not exist. Therefore it is very important to study these sign-free theories numerically, further in detail, in order to find (or exclude) the QCD critical point.

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[4] Strictly speaking 'universality class' is meaningful only in the ε-regime, while the large-N_c limit with fixed quark mass, chemical potential and volume is not in that regime. Therefore the similarity is expected only at qualitative level, similarly to the case of the chiral random model in the planar limit discussed in § 4.1.1. We thank N. Yamamoto, J. Verbaarschot and P. H. Damgaard for discussions on this issue.

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